

Center-symmetric algebras and bialgebras : relevant properties and consequences

Mahouton Norbert Hounkonnou* and Mafoya Landry Dassoundo†

ABSTRACT. Lie admissible algebra structures, called center - symmetric algebras, are defined. Main properties and algebraic consequences are derived and discussed. Bimodules are given and used to build a center - symmetric algebra on the direct sum of underlying vector space and a finite dimensional vector space. Then, the matched pair of center - symmetric algebras is established and related to the matched pair of sub-adjacent Lie algebras. Besides, Manin triple of center - symmetric algebras is defined and linked with their associated matched pairs. Further, center - symmetric bialgebras of center - symmetric algebras are investigated and discussed. Finally, a theorem yielding the equivalence between Manin triple of center - symmetric algebras, matched pairs of Lie algebras and center-symmetric bialgebras is provided.

Keywords. Lie - admissible algebra; Lie algebra; center - symmetric algebra; matched pair; Manin triple; bialgebra; cocycle.

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1. Introduction

Consider the algebra (\mathcal{A}, μ) , i.e., a \mathbb{K} vector space \mathcal{A} endowed with a binary operation or law (bilinear homomorphism) μ defined as:

$$\begin{aligned} \mu : \mathcal{A} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (x, y) &\longmapsto \mu(x, y). \end{aligned}$$

Define the *associator of the binary product* by a trilinear homomorphism on \mathcal{A} as follows [4]:

$$\begin{aligned} ass_\mu : \mathcal{A} \times \mathcal{A} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (x, y, z) &\longmapsto \mu(\mu(x, y), z) - \mu(x, \mu(y, z)). \end{aligned}$$

Let $\sigma \in \Sigma_3$ (symmetry group of degree n ($n \in \mathbb{N}$)), acting on the associator as:

$$\sigma(x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).$$

DEFINITION 1.1. [9] The algebra $\mathcal{A} = (\mathcal{A}, \mu)$ is called *Lie admissible* if the commutator of μ , denoted by $[\cdot, \cdot]_\mu$, defines on \mathcal{A} a Lie algebra structure, i.e., $[x, y]_\mu = \mu(x, y) - \mu(y, x)$ (bilinear and skew-symmetric) and $[[x, y]_\mu, z]_\mu + [[z, x]_\mu, y]_\mu + [[y, z]_\mu, x]_\mu = 0$ (Jacobi identity).

DEFINITION 1.2. [4] The algebra (\mathcal{A}, μ) is called *Lie - admissible* if and only if μ satisfies:

$$\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} \text{ass}_\mu \circ \sigma = 0, \quad (1.1)$$

where ε is the signature of σ .

DEFINITION 1.3. [8] Let G be a subgroup of Σ_3 . We say that the algebra law is G - associative if

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \text{ass}_\mu \circ \sigma = 0. \quad (1.2)$$

The subgroups of Σ_3 are well known. We have: $G_1 = \{\text{id}\}$, $G_2 = \{\text{id}, \tau_{12}\}$, $G_3 = \{\text{id}, \tau_{23}\}$, $G_4 = \{\text{id}, \tau_{13}\}$, $G_5 = \{A_3\}$ (Alternating group) and $G_6 = \Sigma_3$. τ_{ij} is the transposition between i and j , i.e., explicitly:

$$\tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \tau_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \tau_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } \text{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

We deduce the following types of Lie admissible algebras:

- (1) If μ is G_1 - associative, then μ is associative law.
- (2) If μ is G_2 - associative, then μ is a law of Vinberg algebra [12]. If \mathcal{A} is finite - dimensional, then the associated Lie admissible algebra is provided with an affine structure.
- (3) If μ is G_3 - associative, then μ is a law of pre - Lie algebra (also called left - symmetric algebra).
- (4) If μ is G_4 - associative, then μ satisfies

$$(xy)z - x(yz) = (zy)x - z(yx), \quad \forall x, y, z \in \mathcal{A}. \quad (1.3)$$

We called the corresponding algebra *center-symmetric algebra*.

- (5) If μ is G_5 associative, then μ satisfies the generalized Jacobi condition i.e.

$$(xy)z + (yz)x + (zx)y = x(yz) + y(zx) + z(xy). \quad (1.4)$$

Moreover, if the law is antisymmetric, then it is a law of Lie algebra.

- (6) If μ is G_6 - associative, then μ is a Lie admissible Law.

This work aims at studying G_4 - associative structures, called center - symmetric algebras. Their algebraic properties are investigated. Related bimodule and matched pairs are given. Associated Manin triples built look like the Manin triple of Lie algebras [2]. Besides, from symmetry role of matched pairs, equivalent relations are established in the framework of center - symmetric bialgebras making some bridges with the Lie - bialgebra construction by Drinfeld [3].

Throughout this work, we consider \mathcal{A} , a finite dimensional vector space over the field \mathbb{K} of characteristic zero (0) together with a bilinear product \cdot defined as $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $(x, y) \mapsto x \cdot y$.

2. Basic properties: main definitions and algebraic consequences

In this section, we give the definition of the center - symmetric algebra, provide their basic properties and deduce relevant algebraic consequences, similarly to known framework of left - symmetric algebras [1].

DEFINITION 2.1. (\mathcal{A}, \cdot) , (or simply \mathcal{A}), is said to be a *center-symmetric algebra* if $\forall x, y, z \in \mathcal{A}$, the associator of the bilinear product \cdot , defined by $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$, is symmetric in x and z , i.e.,

$$(x, y, z) = (z, y, x). \quad (2.1)$$

As matter of notation simplification, we will denote $x \cdot y$ by xy if not any confusion.

REMARK 2.2. *Any associative algebra is a center-symmetric algebra.*

PROPOSITION 2.3. *The bilinear product (commutator) $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, $(x, y) \longmapsto [x, y] = x \cdot y - y \cdot x$ gives a Lie bracket structure on \mathcal{A} , known as the sub-adjacent Lie algebra $\mathcal{G}(\mathcal{A}) := (\mathcal{A}, [\cdot, \cdot])$ of (\mathcal{A}, \cdot) .*

Proof: By definition of the commutator, $[\cdot, \cdot]$ is bilinear and skew symmetric. The Jacoby identity comes from a straightforward computation. \square

Thus, as in the case of left-symmetric algebras, (\mathcal{A}, \cdot) can be called the compatible center - symmetric algebra structure of the Lie algebra $\mathcal{G}(\mathcal{A})$.

Considering the representations of the left L and right R multiplication operations:

$$\begin{aligned} L : \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ x &\longmapsto L_x : \begin{array}{ccc} \mathcal{A} &\longrightarrow & \mathcal{A} \\ y &\longmapsto & x \cdot y, \end{array} \end{aligned} \quad (2.2)$$

$$\begin{aligned} R : \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ x &\longmapsto R_x : \begin{array}{ccc} \mathcal{A} &\longrightarrow & \mathcal{A} \\ y &\longmapsto & y \cdot x, \end{array} \end{aligned} \quad (2.3)$$

we infer the adjoint representation $\text{ad} := L - R$ of the sub-adjacent Lie algebra $\mathcal{G}(\mathcal{A})$ of a center - symmetric algebra \mathcal{A} as follows:

$$\begin{aligned} \text{ad} : \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ x &\longmapsto \text{ad}_x : \begin{array}{ccc} \mathcal{A} &\longrightarrow & \mathcal{A} \\ y &\longmapsto & [x, y], \end{array} \end{aligned} \quad (2.4)$$

such that $\forall x, y \in \mathcal{A}, \text{ad}_x(y) := (L_x - R_x)(y)$.

PROPOSITION 2.4. *Let (\mathcal{A}, \cdot) be a center - symmetric algebra, and L , (resp. R), be the linear representation of the left, (resp. right), multiplication operator. Then,*

- (1) *For all $x, y \in \mathcal{A}$ we have: $[L_x, R_y] = [L_y, R_x]$ and $L_{x \cdot y} - L_x L_y = R_x R_y - R_{y \cdot x}$.*
- (2) *$\text{ad} = L - R$ is a linear representation of the sub-adjacent Lie algebra $\mathcal{G}(\mathcal{A})$ of (\mathcal{A}, \cdot) , i.e., $\text{ad}_{[x, y]} = [\text{ad}_x, \text{ad}_y]$, $\forall x, y \in \mathcal{A}$.*

Proof: It is immediate from the definitions of considered operators. \square

3. Bimodules and matched pairs

DEFINITION 3.1. *Let \mathcal{A} be a center - symmetric algebra, V be a vector space. Suppose $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$ be two linear maps satisfying: For all $x, y \in \mathcal{A}$,*

$$[l_x, r_y] = [l_y, r_x] \quad (3.1)$$

$$l_{xy} - l_x l_y = r_x r_y - r_{yx}. \quad (3.2)$$

Then, (l, r, V) (or simply (l, r)) is called bimodule of the center - symmetric algebra \mathcal{A} .

In this case, the following statement can be proved by a direct computation.

PROPOSITION 3.2. *Let (\mathcal{A}, \cdot) be a center-symmetric algebra and V be a vector space over \mathbb{K} . Consider two linear maps, $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of \mathcal{A} if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into a center-symmetric algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1)$, $\forall x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$. We denote it by $\mathcal{A} \ltimes_{l, r} V$ or simply $\mathcal{A} \ltimes V$.*

Furthermore, we derive the next result.

PROPOSITION 3.3. *Let \mathcal{A} be a center-symmetric algebra and V be a vector space over \mathbb{K} . Consider two linear maps, $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$, such that (l, r, V) is a bimodule of \mathcal{A} . Then, the map: $l - r : \mathcal{A} \longrightarrow \mathfrak{gl}(V)$ $x \longmapsto l_x - r_x$, is a linear representation of the sub-adjacent Lie algebra of \mathcal{A} .*

EXAMPLE 3.4. According to the Proposition 2.4, one can deduce that (L, R, V) is a bimodule of the center-symmetric algebra \mathcal{A} , where L and R are the left and right multiplication operator representations, respectively.

DEFINITION 3.5. [6] Let \mathcal{G} and \mathcal{H} be two Lie algebras and let $\mu : \mathcal{H} \rightarrow \mathfrak{gl}(\mathcal{G})$ and $\rho : \mathcal{G} \rightarrow \mathfrak{gl}(\mathcal{H})$ be two Lie algebra representations satisfying: For all $x, y \in \mathcal{G}, a, b \in \mathcal{H}$,

$$\rho(x)[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0, \quad (3.3)$$

$$\mu(a)[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \quad (3.4)$$

Then, $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ is called a matched pair of the Lie algebras \mathcal{G} and \mathcal{H} , denoted by $\mathcal{H} \bowtie_{\mu}^{\rho} \mathcal{G}$. In this case, $(\mathcal{G} \oplus \mathcal{H}, *)$ defines a Lie algebra with respect to the product $*$ satisfying:

$$(x + a) * (y + b) = [x, y] + \mu(a)y - \mu(b)x + [a, b] + \rho(x)b - \rho(y)a.$$

THEOREM 3.6. Let (\mathcal{A}, \cdot) and (\mathcal{B}, \circ) be two center - symmetric algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of \mathcal{A} and \mathcal{B} , respectively, obeying the relations:

$$-r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + a \circ (r_{\mathcal{A}}(x)b) + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + (l_{\mathcal{A}}(x)b) \circ a - l_{\mathcal{A}}(x)(b \circ a) = 0, \quad (3.5)$$

$$-r_{\mathcal{B}}(a)(x \cdot y) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + x \cdot (r_{\mathcal{B}}(a)y) + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + (l_{\mathcal{B}}(a)y) \cdot x - l_{\mathcal{B}}(a)(y \cdot x) = 0, \quad (3.6)$$

$$a \circ (l_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a - (r_{\mathcal{A}}(x)a) \circ b - l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + \\ -b \circ (l_{\mathcal{A}}(x)a) - r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b = 0, \quad (3.7)$$

$$x \cdot (l_{\mathcal{B}}(a)y) + (r_{\mathcal{B}}(a)y) \cdot x - (r_{\mathcal{B}}(a)x) \cdot y - l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + \\ -y \cdot (l_{\mathcal{B}}(a)x) - r_{\mathcal{B}}(r_{\mathcal{A}}(a)x)y = 0, \quad (3.8)$$

for any $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$. Then, there is a center - symmetric algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by: $(x + a) * (y + b) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)$. We denote this center - symmetric algebra by $\mathcal{A} \bowtie_{l_{\mathcal{B}}, r_{\mathcal{B}}}^{l_{\mathcal{A}}, r_{\mathcal{A}}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie \mathcal{B}$. Then $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ satisfying the above conditions is called matched pair of the center - symmetric algebras \mathcal{A} and \mathcal{B} .

Proof: Consider $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$.

$$\text{We have } (x + a) * (y + b) = (xy + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a),$$

and the associator takes the form:

$$(x + a, y + b, z + c) = (x, y, z) + (a, b, c) + \{r_{\mathcal{B}}(c)(x \cdot y) + l_{\mathcal{A}}(x \cdot y)c - x \cdot (r_{\mathcal{B}}(c)y) \\ - l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c) - r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x\} + \{r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x) + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c \\ - r_{\mathcal{B}}(b \circ c)x + (l_{\mathcal{A}}(x)b) \circ c - l_{\mathcal{A}}(x)(b \circ c)\} + \{(r_{\mathcal{B}}(b)x) \cdot z + l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z \\ + r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b) - x \cdot (l_{\mathcal{B}}(b)z) - r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x - l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)\} + \\ \{(l_{\mathcal{B}}(a)y) \cdot z + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z + r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a) - l_{\mathcal{B}}(a)(y \cdot z) - r_{\mathcal{A}}(y \cdot z)a\} \\ + \{r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y) + (r_{\mathcal{A}}(y)a) \circ c + l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c - l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y) \\ - a \circ (l_{\mathcal{A}}(y)c) - r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\} + \{l_{\mathcal{B}}(a \circ b)z + r_{\mathcal{A}}(z)(a \circ b) - \\ + l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z) - a \circ (r_{\mathcal{A}}(z)b) - r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\},$$

which can also be re-expressed as:

$$(x + a, y + b, z + c) = (x, y, z) + (x, y, c) + (x, b, z) + (x, b, c) \\ + (a, y, z) + (a, y, c) + (a, b, z) + (a, b, c). \quad (3.9)$$

Similarly,

$$(z + c, y + c, x + a) = (z, y, x) + (z, y, a) + (z, b, x) + (z, b, a) \\ + (c, y, x) + (c, b, a) + (c, y, a) + (c, b, x). \quad (3.10)$$

Using the fact that $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of \mathcal{A} and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of \mathcal{B} , one arrives at the following result:

$$(x + a, y + b, z + c) = (z + c, y + b, x + a) \iff \begin{cases} (x, y, z) = (z, y, x) \\ (x, y, c) = (c, y, x) \\ (x, b, z) = (z, b, x) \\ (x, b, c) = (c, b, x) \\ (a, y, z) = (z, y, a) \\ (a, y, c) = (c, y, a) \\ (a, b, z) = (z, b, a) \\ (a, b, c) = (c, b, a) \end{cases} \iff \begin{cases} (l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B}) & , & (l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A}) \\ (x, y, c) & = & (c, y, x) \quad (3.6) \\ (x, b, z) & = & (z, b, x) \quad (3.8) \\ (x, b, c) & = & (c, b, x) \quad (3.5) \\ (a, y, c) & = & (c, y, a). \quad (3.7) \end{cases}$$

This last relation ends the proof. \square

Moreover, every center - symmetric algebra which is a direct sum of the underlying spaces of two center - symmetric sub - algebras can be obtained in the above way.

COROLLARY 3.7. *Let $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ be a matched pair of center - symmetric algebras. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), l_{\mathcal{A}} - r_{\mathcal{A}}, l_{\mathcal{B}} - r_{\mathcal{B}})$ is a matched pair of sub-adjacent Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$.*

Proof: By using the Proposition 3.3 and the bimodules $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$, we have: $\text{ad}_{\mathcal{A}} := l_{\mathcal{A}} - r_{\mathcal{A}}$ and $\text{ad}_{\mathcal{B}} := l_{\mathcal{B}} - r_{\mathcal{B}}$ are the linear representations of the sub-adjacent Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of the center-symmetric algebras \mathcal{A} and \mathcal{B} , respectively. Then, the statement that $\mathcal{G}(\mathcal{A}) \bowtie_{\text{ad}_{\mathcal{B}}}^{\text{ad}_{\mathcal{A}}} \mathcal{G}(\mathcal{B})$ is a matched pair of the Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ follows from Theorem 3.6. Hence, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), \text{ad}_{\mathcal{A}}, \text{ad}_{\mathcal{B}})$ is a matched pair of sub-adjacent Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$. \square

DEFINITION 3.8. *Let (l, r, V) be a bimodule of a center - symmetric algebra \mathcal{A} , where V is a finite dimensional vector space. The dual maps l^*, r^* of the linear maps l, r , are defined, respectively, as: $l^*, r^* : \mathcal{A} \rightarrow \mathfrak{gl}(V^*)$ such that:*

$$\begin{array}{ccc} l^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ & & V^* \longrightarrow V^* \\ x \longmapsto l_x^* : u^* & \longmapsto & l_x^* u^* : \begin{array}{ccc} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle l_x^* u^*, v \rangle := \langle u^*, l_x v \rangle, \end{array} \end{array} \quad (3.11)$$

$$\begin{array}{ccc} r^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ & & V^* \longrightarrow V^* \\ x \longmapsto l_x^* : u^* & \longmapsto & r_x^* u^* : \begin{array}{ccc} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle r_x^* u^*, v \rangle := \langle u^*, r_x v \rangle, \end{array} \end{array} \quad (3.12)$$

for all $x \in \mathcal{A}, u^* \in V^*, v \in V$.

PROPOSITION 3.9. *Let \mathcal{A} be a center - symmetric algebra and $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$ be two linear maps, where V is a finite dimensional vector space. The following conditions are equivalent:*

- (1) (l, r, V) is a bimodule of \mathcal{A} .
- (2) (r^*, l^*, V^*) is a bimodule of \mathcal{A} .

Proof: It stems from the Definition 3.8.

THEOREM 3.10. *Let (\mathcal{A}, \cdot) be a center - symmetric algebra. Suppose that there exists a center - symmetric algebra structure " \circ " on its dual space \mathcal{A}^* . Then, $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_{\circ}^*, L_{\circ}^*)$ is a matched pair of center - symmetric algebras \mathcal{A} and \mathcal{A}^* if and only if $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\text{ad}^*, -\text{ad}_{\circ}^*)$ is a matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$.*

Proof: By considering the Theorem 3.6, setting $l_{\mathcal{A}} := R^*, r_{\mathcal{A}} := L^*, l_{\mathcal{B}} := R_{\circ}^*, r_{\mathcal{B}} := L_{\circ}^*$, and exploiting the Definition 3.5 with $\mathcal{G} := \mathcal{G}(\mathcal{A}), \mathcal{H} := \mathcal{G}(\mathcal{A}^*), \rho := R^* - L^*, \mu := R_{\circ}^* - L_{\circ}^*$, and the relations (3.11) and (3.12), we get the equivalences. \square

PROPOSITION 3.11. *Let \mathcal{G} be a Lie algebra. Suppose $\rho : \mathcal{G} \rightarrow \mathfrak{gl}(V)$ and $\mu : \mathcal{G} \rightarrow \mathfrak{gl}(W)$ be two linear representations of \mathcal{G} , where V and W are two vector spaces. Then, the linear map $\rho \otimes 1 + 1 \otimes \mu : \mathcal{G} \rightarrow \mathfrak{gl}(V \otimes W)$ given by $(\rho \otimes 1 + 1 \otimes \mu)(v, w) := \rho(x)v \otimes w + v \otimes \mu(x)v$ is also a representation of \mathcal{G} .*

Proof: It comes from a straightforward computation. \square

THEOREM 3.12. *Let \mathcal{A} be a center - symmetric algebra with the product given by the linear map $\beta^* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$. Suppose there is a center - symmetric algebra structure " \circ " on the dual space \mathcal{A}^* provided by a linear map $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\text{ad}^*, -\text{ad}_{\circ}^*)$ is a matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$ if and only if $\alpha : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is a 1 cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\text{ad}_{\circ}) \otimes 1 + 1 \otimes (-\text{ad}_{\circ})$ and $\beta : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ is a 1-cocycle of $\mathcal{G}(\mathcal{A}^*)$ associated to $(-\text{ad}_{\circ}) \otimes 1 + 1 \otimes (-\text{ad}_{\circ})$.*

Proof: See Appendix.

4. Manin triple and center-symmetric bialgebras

In this section, similarly to the notion of Manin triple of Lie algebras introduced in [2], we first give the definition of Manin triple of a center - symmetric algebra and investigate its associated bialgebra structure. Then, we provide the basic definition and properties of center - symmetric bialgebras.

DEFINITION 4.1. *A Manin triple of center - symmetric algebras is a triple $(\mathcal{A}, \mathcal{A}^+, \mathcal{A}^-)$ together with a nondegenerate symmetric bilinear form $\mathfrak{B}(\cdot, \cdot)$ on \mathcal{A} which is invariant, i.e., $\forall x, y, z \in \mathcal{A}, \mathfrak{B}(x * y, z) = \mathfrak{B}(x, y * z)$, satisfying:*

- (1) $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ as \mathbb{K} - vector space;
- (2) \mathcal{A}^+ and \mathcal{A}^- are center - symmetric subalgebras of \mathcal{A} ;
- (3) \mathcal{A}^+ and \mathcal{A}^- are isotropic with respect to $\mathfrak{B}(\cdot, \cdot)$, i.e., $\mathfrak{B}(\mathcal{A}^+; \mathcal{A}^+) = 0 = \mathfrak{B}(\mathcal{A}^-; \mathcal{A}^-)$.

Two Manin triples $(\mathcal{A}_1, \mathcal{A}_1^+, \mathcal{A}_1^-, \mathfrak{B}_1)$ and $(\mathcal{A}_2, \mathcal{A}_2^+, \mathcal{A}_2^-, \mathfrak{B}_2)$ of center - symmetric algebras \mathcal{A}_1 and \mathcal{A}_2 are homomorphic (isomorphic) if there is a homomorphism (isomorphism) $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that: $\varphi(\mathcal{A}_1^+) \subset \mathcal{A}_2^+, \varphi(\mathcal{A}_1^-) \subset \mathcal{A}_2^-, \mathfrak{B}_1(x, y) = \varphi^* \mathfrak{B}_2(\varphi(x), \varphi(y)) = \mathfrak{B}_2(\varphi(x), \varphi(y))$. In particular, if (\mathcal{A}, \cdot) is a center - symmetric algebra, and if there exists a center - symmetric algebra structure on its dual space \mathcal{A}^* denoted (\mathcal{A}^*, \circ) , then there is a center - symmetric algebra structure on the direct sum of the underlying vector space of \mathcal{A} and \mathcal{A}^* (see Theorem 3.6) such that $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the associated Manin triple with the invariant bilinear symmetric form given by $\mathfrak{B}_{\mathcal{A}}(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle y, a^* \rangle, \forall x, y \in \mathcal{A}; a^*, b^* \in \mathcal{A}^*$, called the standard Manin triple of the center - symmetric algebra \mathcal{A} .

THEOREM 4.2. *Let (\mathcal{A}, \cdot) and (\mathcal{A}^*, \circ) be two center - symmetric algebras. Then, the sextuple $(\mathcal{A}, \mathcal{A}^*, R^*, L^*; R_{\circ}^*, L_{\circ}^*)$ is a matched pair of center - symmetric algebras \mathcal{A} and \mathcal{A}^* if and only if $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is their standard Manin triple.*

Proof: By considering that $(\mathcal{A}, \mathcal{A}^*, R^*, L^*; R_{\circ}^*, L_{\circ}^*)$ is a matched pair of center - symmetric algebras, it follows that the bilinear product $*$ defined in the Theorem 3.6 is center - symmetric on the direct sum of underlying vectors spaces, $\mathcal{A} \oplus \mathcal{A}^*$. Computing and comparing the relations, we get: $\mathfrak{B}_{\mathcal{A}}((x + a) * (y + b), (z + c)) = \mathfrak{B}_{\mathcal{A}}((x + a), (y + b) * (z + c)) \forall x, y, z \in \mathcal{A}; a, b, c \in \mathcal{A}^*$, which expresses the invariance of the standard bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$. Therefore, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of the center - symmetric algebras \mathcal{A} and \mathcal{A}^* . \square

DEFINITION 4.3. *Let \mathcal{A} be a vector space. A center - symmetric bialgebra structure on \mathcal{A} is a pair of linear maps (α, β) such that $\alpha : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \beta : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ satisfying:*

- (1) $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$ is a center - symmetric algebra structure on \mathcal{A}^* ,

- (2) $\beta^* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is a center - symmetric algebra structure on \mathcal{A} ,
- (3) α is a 1 cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\text{ad.}) \otimes 1 + 1 \otimes (-\text{ad.})$,
- (4) β is 1-cocycle of $\mathcal{G}(\mathcal{A}^*)$ associated to $(-\text{ad}_o) \otimes 1 + 1 \otimes (-\text{ad}_o)$.

We also denote this center - symmetric bialgebra by $(\mathcal{A}, \mathcal{A}^*, \alpha, \beta)$ or simply $(\mathcal{A}, \mathcal{A}^*)$.

PROPOSITION 4.4. *Let (\mathcal{A}, \cdot) be a center - symmetric algebra and (\mathcal{A}^*, \circ) be a center - symmetric algebra structure on its dual space \mathcal{A}^* . Then the following conditions are equivalent:*

- (1) $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of considered center - symmetric algebras;
- (2) $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\text{ad}^*, -\text{ad}_o^*)$ is a matched pair of sub-adjacent Lie algebras;
- (3) $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_o^*, L_o^*)$ is a matched pair of center - symmetric algebras;
- (4) $(\mathcal{A}, \mathcal{A}^*)$ is a center -symmetric bialgebra.

Proof: From Theorem 3.10, (2) \iff (3), while from Theorem 3.12, (2) \iff (4). Theorem 4.2 shows that (1) \iff (3). \square

5. Concluding remarks

In this work, we have defined Lie admissible algebra structures, called center - symmetric algebras for which main properties and algebraic consequences have been derived and discussed. Bimodules have been given and used to build a center - symmetric algebra on the direct sum of a center - symmetric algebra and a vector space. Then, we have established the matched pair of center - symmetric algebras, which has been related to the matched pair of sub - adjacent Lie algebras. Besides, we have defined the Manin triple of center - symmetric algebras and linked it with their associated matched pairs. Further, we have investigated and discussed center - symmetric bialgebras of center - symmetric algebras. Finally, we have provided a theorem yielding the equivalence between Manin triple of center - symmetric algebras, matched pairs of Lie algebras and center - symmetric algebras, and center - symmetric bialgebra.

Appendix

Proof of the Theorem 3.12

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of \mathcal{A} and $\{e_1^*, e_2^*, \dots, e_n^*\}$ its dual basis. Consider $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$

and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$, where $c_{ij}^k, f_{ij}^k \in \mathbb{K}$ are structure constants associated to \cdot and \circ , respectively.

Then, it follows that:

$$\alpha(e_k) = \sum_{i,j=1}^n f_{ij}^k e_i \otimes e_j, \quad \beta(e_k^*) = \sum_{i,j=1}^n c_{ij}^k e_i^* \otimes e_j^*, \quad \text{and}$$

$$\alpha([e_i, e_j]) = \sum_{m,l=1}^n \sum_{k=1}^n \{(c_{ij}^k - c_{ji}^k) f_{ml}^k\} e_m \otimes e_l, \quad (5.1)$$

$$\beta([e_i^*, e_j^*]) = \sum_{m,l=1}^n \sum_{k=1}^n \{(f_{ij}^k - f_{ji}^k) c_{ml}^k\} e_m^* \otimes e_l^*, \quad (5.2)$$

and we get:

$$\{(-\text{ad.})(e_i) \otimes 1 + 1 \otimes (-\text{ad.})(e_i)\} \alpha(e_j) - \{(-\text{ad.})(e_j) \otimes 1 + 1 \otimes (-\text{ad.})(e_j)\} \alpha(e_i) =$$

$$\sum_{m,l=1}^n \sum_{k=1}^n \left\{ -f_{kl}^j (c_{ik}^m - c_{ki}^m) + f_{kl}^i (c_{jk}^m - c_{kj}^m) - f_{mk}^j (c_{il}^m - c_{li}^m) + f_{mk}^i (c_{jl}^m - c_{lj}^m) \right\} e_m \otimes e_l \quad (5.3)$$

Taking into account the fact that α is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\text{ad.}) \otimes 1 + 1 \otimes (-\text{ad.})$, and using the relations (5.1) and (5.3) yield:

$$\sum_{k=1}^n (c_{ij}^k - c_{ji}^k) f_{ml}^k = \sum_{k=1}^n \left\{ f_{kl}^i (c_{jk}^m - c_{kj}^m) - f_{kl}^j (c_{ik}^m - c_{ki}^m) + f_{mk}^i (c_{jl}^m - c_{lj}^m) - f_{mk}^j (c_{il}^m - c_{li}^m) \right\}. \quad (5.4)$$

Besides, we obtain:

$$\{(-\text{ad}_o)(e_i^*) \otimes 1 + 1 \otimes (-\text{ad}_o)(e_i^*)\} \beta(e_j^*) - \{(-\text{ad}_o)(e_j^*) \otimes 1 + 1 \otimes (-\text{ad}_o)(e_j^*)\} \beta(e_i^*) = \sum_{m,l=1}^n \sum_{k=1}^n \left\{ -c_{kl}^j (f_{ik}^m - f_{ki}^m) + c_{kl}^i (f_{jk}^m - f_{kj}^m) - c_{mk}^j (f_{ik}^l - f_{ki}^l) + c_{mk}^i (f_{jk}^l - f_{kj}^l) \right\} (e_m^* \otimes e_l^*). \quad (5.5)$$

As β is the 1-cocycle issued from $(-\text{ad}_o) \otimes 1 + 1 \otimes (-\text{ad}_o)$ and using the relations (5.2) and (5.5), we obtain:

$$\sum_{k=1}^n (f_{ij}^k - f_{ji}^k) c_{ml}^k = \sum_{k=1}^n \left\{ c_{kl}^i (f_{jk}^m - f_{kj}^m) - c_{kl}^j (f_{ik}^m - f_{ki}^m) + c_{mk}^i (f_{jk}^l - f_{kj}^l) - c_{mk}^j (f_{ik}^l - f_{ki}^l) \right\}. \quad (5.6)$$

Now, let us find the relations associated to the equations (3.3) - (3.4) of the matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$. We have $\forall i, j, k$:

$$\langle (-\text{ad}^*)(e_i) e_j^*, e_k \rangle = - \left\langle \sum_{k=1}^m (c_{ik}^j - c_{ki}^j) e_k^*, e_k \right\rangle,$$

providing

$$(-\text{ad}^*)(e_i) e_j^* = - \sum_{k=1}^n (c_{ik}^j - c_{ki}^j) e_k^*. \quad (5.7)$$

Similarly,

$$(-\text{ad}_o^*)(e_i^*) e_j = - \sum_{k=1}^n (f_{ik}^j - f_{ki}^j) e_k, \quad (5.8)$$

$$(-\text{ad}_o^*)(e_m^*) [e_i, e_j] = \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) (-\text{ad}_o^*)(e_m^*) e_k - \sum_{l=1}^n \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) (f_{ml}^k - f_{lm}^k) e_l.$$

Then,

$$\begin{aligned} (-\text{ad}_o^*)(e_m^*) [e_i, e_j] &= - \sum_{l=1}^n \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) (f_{ml}^k - f_{lm}^k) e_l, \\ &= -\text{ad}_o^*(\text{ad}^*(e_i) e_m^*) e_j - [e_i, \text{ad}_o^*(e_m^*) e_j] + \text{ad}_o^*(\text{ad}^*(e_j) e_m^*) - [\text{ad}_o^*(e_m^*) e_i, e_j] \\ &= \sum_{l=1}^n \sum_{k=1}^n \{ -(c_{ik}^m - c_{ki}^m) (f_{kl}^j - f_{lk}^j) - (f_{mk}^j - f_{km}^j) (c_{ik}^l - c_{ki}^l) + \\ &\quad (c_{jk}^m - c_{kj}^m) (f_{kl}^i - f_{lk}^i) - (f_{mk}^i - f_{km}^i) (c_{kj}^l - c_{jk}^l) \} e_l. \end{aligned} \quad (5.9)$$

Using the fact that $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), \text{ad}^*, \text{ad}_o^*)$ is a bimodule of Lie algebras, we have

$$\begin{aligned} \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) (f_{ml}^k - f_{lm}^k) &= \sum_{k=1}^n -(c_{ik}^m - c_{ki}^m) (f_{kl}^j - f_{lk}^j) - (f_{mk}^j - f_{km}^j) (c_{ik}^l - c_{ki}^l) + \\ &\quad (c_{jk}^m - c_{kj}^m) (f_{kl}^i - f_{lk}^i) + (f_{mk}^i - f_{km}^i) (c_{kj}^l - c_{jk}^l), \end{aligned} \quad (5.10)$$

that is,

$$\begin{aligned} \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) f_{ml}^k + \sum_{k=1}^n (c_{ik}^m - c_{ki}^m) f_{kl}^j + (c_{ik}^l - c_{ki}^l) f_{mk}^j - (c_{jk}^m - c_{kj}^m) f_{kl}^i - (c_{jk}^l - c_{kj}^l) f_{mk}^i = \\ \sum_{k=1}^n (c_{ij}^k - c_{ji}^k) f_{lm}^k + \sum_{k=1}^n (c_{ik}^m - c_{ki}^m) f_{lk}^j + (c_{ik}^l - c_{ki}^l) f_{km}^j - (c_{jk}^m - c_{kj}^m) f_{lk}^i - (c_{jk}^l - c_{kj}^l) f_{km}^i. \end{aligned}$$

Replacing l (resp. m) by m (resp. l) in the right-hand side of the equality leads to:

$$\sum_{k=1}^n (c_{ij}^k - c_{ji}^k) f_{ml}^k = \sum_{k=1}^n \{ -(c_{ik}^m - c_{ki}^m) f_{kl}^j - (c_{ik}^l - c_{ki}^l) f_{mk}^j + (c_{jk}^m - c_{kj}^m) f_{kl}^i + (c_{jk}^l - c_{kj}^l) f_{mk}^i \} \quad (5.11)$$

which is identical to the equation (5.4). Besides,

$$\begin{aligned}
 (-\text{ad}^*)(e_m)[e_i^*, e_j^*] &= - \sum_{l=1}^n \sum_{k=1}^n \{ (f_{ij}^k - f_{ji}^k)(c_{ml}^k - c_{lm}^k) \} e_l^* \\
 &\quad - \text{ad}^*(\text{ad}_o^*(e_i^*)e_m)e_j^* - [e_i^*, \text{ad}^*(e_m)e_j^*] + \text{ad}^*(\text{ad}_o^*(e_j^*)e_m)e_i^* - [\text{ad}^*(e_m)e_i^*, e_j^*] = \\
 &\quad \sum_{l=1}^n \sum_{k=1}^n \{ -(f_{ik}^m - f_{ki}^m)(c_{kl}^j - c_{lk}^j) - (c_{mk}^j - c_{km}^j)(f_{ik}^l - f_{ki}^l) + (f_{jk}^m - f_{kj}^m)(c_{kl}^i - c_{lk}^i) - \\
 &\quad (c_{mk}^i - c_{km}^i)(f_{kj}^l - f_{jk}^l) \} e_l^*.
 \end{aligned} \tag{5.12}$$

Then, with $\mathcal{G}(\mathcal{A}) \bowtie_{-\text{ad}_o^*}^{-\text{ad}^*} \mathcal{G}(\mathcal{A}^*)$ and the relation (5.12), we obtain

$$\begin{aligned}
 \sum_{k=1}^n (f_{ij}^k - f_{ji}^k)(c_{ml}^k - c_{lm}^k) &= \sum_{k=1}^n -(f_{ik}^m - f_{ki}^m)(c_{kl}^j - c_{lk}^j) - (c_{mk}^j - c_{km}^j)(f_{ik}^l - f_{ki}^l) + \\
 &\quad + (f_{jk}^m - f_{kj}^m)(c_{kl}^i - c_{lk}^i) + (c_{mk}^i - c_{km}^i)(f_{jk}^l - f_{kj}^l),
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 \sum_{k=1}^n (f_{ij}^k - f_{ji}^k)c_{ml}^k + \sum_{k=1}^n c_{kl}^j(f_{ik}^m - f_{ki}^m) + c_{kl}^i(f_{jk}^m - f_{kj}^m) - c_{mk}^j(f_{ik}^l - f_{ki}^l) - c_{mk}^i(f_{jk}^l - f_{kj}^l) = \\
 \sum_{k=1}^n (f_{ij}^k - f_{ji}^k)c_{lm}^k + \sum_{k=1}^n c_{lk}^j(f_{ik}^m - f_{ki}^m) + c_{lk}^i(f_{jk}^m - f_{kj}^m) - c_{km}^j(f_{ik}^l - f_{ki}^l) - c_{km}^i(f_{jk}^l - f_{kj}^l),
 \end{aligned}$$

Replacing l , (resp. m), by m , (resp. l), in the right-hand side of the equality leads to

$$\sum_{k=1}^n (f_{ij}^k - f_{ji}^k)c_{ml}^k = \sum_{k=1}^n -c_{kl}^j(f_{ik}^m - f_{ki}^m) + c_{kl}^i(f_{jk}^m - f_{kj}^m) - c_{mk}^j(f_{ik}^l - f_{ki}^l) + c_{mk}^i(f_{jk}^l - f_{kj}^l), \tag{5.13}$$

which is identical to the equation (5.6). \square

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(*) UNIVERSITY OF ABOMEY-CALAVI, INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS,
ICMPA-UNESCO CHAIR, 072 BP 50, COTONOU, REP. OF BENIN

E-mail address: `norbert.hounkonnou@cipma.uac.bj`, with copy to `hounkonnou@yahoo.fr`

(†) UNIVERSITY OF ABOMEY-CALAVI, INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS,
ICMPA-UNESCO CHAIR, 072 BP 50, COTONOU, REP. OF BENIN

E-mail address: `mafoya.dassoundo@cipma.uac.bj`, with copy to `maflandas@gmail.com`